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Rational Approximations to e^x

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We obtain an upper bound on how well e^x can be approximated on $[-1, 1]$ by (m, n)-degree rational functions (i.e., rational functions whose numerator has degree m and whose denominator has degree n). In Meinardus' monograph [1, p. 168] it is shown that, at least when $n = 1$, the (m, n) degree rationals do asymptotically better than the $(m + n)$-degree polynomials.

He makes the conjecture that the (sup-norm) distance from e^x to the space of (m, n) degree rationals is asymptotically

$$\frac{m!\, n!}{2^{m+n}(m+n)!\,(m+n+1)!} \quad \text{as} \quad m + n \to \infty$$

and points out that H. Werner has obtained related numerical evidence. We prove that this quantity (multiplied by a constant factor) does serve as an upper bound for this proximity.

Set $R(z) = \int_0^\infty t^n(t+z)^m e^{-t}\, dt / \int_0^\infty (t-z)^n t^m e^{-t}\, dt$. ($R(z)$ is, in fact, the (m, n)-degree Padé approximant to e^z.) Note that

$$\int_0^\infty (t-z)^n t^m e^{-t}\, dt(e^z - R(z))$$

$$= \int_0^\infty (t-z)^n t^m e^{z-t}\, dt - \int_0^\infty t^n(t+z)^m e^{-t}\, dt$$

$$= \int_0^\infty (t-z)^n t^m e^{z-t}\, dt - \int_z^\infty (t-z)^n t^m e^{z-t}\, dt$$

$$= \int_0^z (t-z)^n t^m e^{z-t}\, dt = z^{m+n+1} \int_0^1 (u-1)^n u^m e^{(1-u)z}\, dt$$

$$\ll |z|^{m+n+1} e^{|z|} \int_0^1 (1-u)^n u^m\, du = \frac{|z|^{m+n+1}}{(m+n+1)!} e^{|z|}\, m!\, n!\,.$$

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Also, if $|z| = \frac{1}{2}$, then

$$\int_0^\infty (t-z)^n t^m e^{-t}\,dt \geqslant \int_0^\infty \left(t^n - \binom{n}{1}\frac{t^{n-1}}{2} - \binom{n}{2}\frac{t^{n-2}}{4} - \cdots\right) t^m e^{-t}\,dt$$

$$\geqslant 2\int_0^\infty t^{n+m}e^{-t}\,dt - \int_0^\infty \left(t+\frac{1}{2}\right)^n t^m e^{-t}\,dt$$

$$\geqslant 2(n+m)! - \int_{-1/2}^\infty \left(t+\frac{1}{2}\right)^{n+m} e^{-t}\,dt$$

$$= (2 - e^{1/2})(n+m)!\,.$$

Altogether, then, we have

$$|e^z - R(z)| \leqslant \frac{e^{1/2}}{2-e^{1/2}}\,\frac{2^{-m-n-1}\,m!\,n!}{(m+n)!\,(m+n+1)!} \qquad \text{in} \quad |z| \leqslant \frac{1}{2}.$$

If we now write $z = (x + iy)/2$, with $x^2 + y^2 = 1$, we obtain the above bound for both $e^z - R(z)$ and $e^{\bar{z}} - R(\bar{z})$, and multiplication yields thereby

$$|e^x - R(z)\,R(\bar{z})| \leqslant 8\,\frac{2^{-m-n}\,m!\,n!}{(m+n)!\,(m+n+1)!}\,.$$

Our proof is completed by the observation that $R(z)\,R(\bar{z})$ is an (m, n)-degree rational function in the variable x.

Reference

1. G. Meinardus, "Aproximation of Functions: Theory and Numerical Methods," Springer-Verlag, New York, 1967.